MATHEMATICS SPECIALIST 3CD
COMMON TEST 5 – Term 3 2010

Topic(s): *Rectilinear Motion*
Simple Harmonic Motion
Mathematical Reasoning
Marginal Analysis
Exponential Growth/Decay

SOLUTIONS

Name:____________________________ Marks:_________ / 50

Instructions:

- Answer all the questions in the spaces provided
- Casio Classpad Calculator may be used
- External notes are not allowed
- Duration of test: 50 minutes
- This test contributes to 5% of the year (school) mark

1. **[8 marks]**

Prove by mathematical induction that

$$1 + 4 + 7 + 10 + \ldots + (3n-2) = \frac{n(3n-1)}{2}$$

Test for $n=1$:

$LHS = 1$

$RHS = \frac{1(3-1)}{2} = 1$

$\therefore$ True for $n=1$

Assume true for $n=k$:

$$1 + 4 + 7 + \ldots + (3k-2) = \frac{k(3k-1)}{2}$$

RTP: $1 + 4 + 7 + \ldots + (3k-2) + (3(k+1)-2) = \frac{(k+1)(3(k+1)-1)}{2}$

$= \frac{(k+1)(3k+2)}{2}$

$$LHS = \frac{k(3k-1)}{2} + (3(k+1)-2)$$

$$= \frac{3k^2-k}{2} + \frac{6k+2}{2}$$

$$= \frac{3k^2+5k+2}{2}$$

$$= \frac{(k+1)(3k+2)}{2}$$

$$= RHS$$

✓ Show true for $n=1$
✓ Assume true for $n=k$
✓ Express for $n=k+1$
✓ Relate to sum till k
✓✓ Use of Algebra
✓ Show when $n=k+1$ (factors)
✓ True for all n

$\therefore$ True for $n=k+1$

$\Rightarrow$ true for all n (integers ≥ 1)

8

2. **[8 marks]**

A company manufactures action figures. It marketing department determines the price-demand function and the cost function defined below:

$p(x) = 119 - 6x \quad , \quad for \quad 1 \le x \le 15$

$C(x) = 234 + 23x$

Where p is the wholesale price per action figure at which x million action figures can be sold, and C is in millions of dollars.

(a) Determine the revenue and profit functions, $R(x)$ and $P(x)$.

2m

$R(x) = \underline{119x - 6x^2}$ ✓

$\therefore \; P(x) = 119x - 6x^2 - 234 - 23x$

$= \underline{96x - 6x^2 - 234}$ ✓

[2]

(b) Verify that $C'(8) = R'(8)$, and state the significance of this result.

2m

$C'(8) = 23 \quad , \quad R'(8) = 23$

When 8 million action figures are sold, the marginal profit is ZERO.

✓ verifty

✓ comment

[2]

(c) Explain why we can approximate the cost of producing the $(x+1)^{\text{th}}$ item using the marginal cost function $C'(x) \approx C(x+1) - C(x)$.

2m

$C(x+1) - C(x) = \dfrac{C(x+1) - C(x)}{1}$ which is the gradient of the straight line connecting $C(x)$ to $C(x+1)$.

$C'(x)$ is the first derivative of $C(x)$ at x, which is approximated by the limit of the gradient of the str. line above, as $C(x+1)$ approaches $C(x)$.

✓ gradient

✓ "limiting chord"

[2]

(d) Find the maximum profit from the sale of the action figures.

2m

From calculator: Max profit of "150" occurs when "$x = 8$".

$\therefore$ Max. profit is $\underline{\$150,000,000}$

✓ use of calculator or calculus

✓ correct profit

[2]

8

3. **[12 marks]**

According to Newton's law of Cooling, the temperature T°C (Celsius) of a hot metal slab left to cool down satisfies the equation

$$\frac{dT}{dt} = -k(T-20)$$

where k is a positive constant and t is measured in minutes.

(a) In the expression provided above, what does the number 20 represent?

1m

The ambient or room temperature is $20^\circ C$. ✓

[1]

(b) After 20 minutes the temperature of the slab is 50° C and after 40 minutes it is 30° C. Use the method of separation of variables to determine T as a function of t.

8m

$$\frac{dT}{T-20} = -k\,dt$$

$$\int \frac{dT}{T-20} = \int -k\,dt$$

✓ Separate variables.
✓ Correct integration.
✓✓ T as an exponential function
✓✓ Solve for k.
✓ Solve for A.
✓ Correct function produced.

$$\ln(T-20) = kt + c$$

$$T-20 = e^{kt+c}$$

$$T = e^{c}\cdot e^{kt} + 20$$

$$T = Ae^{kt} + 20$$

sub $t=20$, $T=50$: $50 = Ae^{20k} + 20 \Rightarrow Ae^{20k} = 30$ —— ①

sub $t=40$, $T=30$: $30 = Ae^{40k} + 20. \Rightarrow Ae^{40k} = 10$ —— ②

① ÷ ②: $e^{20k} = \frac{1}{3}$

$\Rightarrow k = \frac{-\ln 3}{20} = -0.05493\ldots$

sub $k = \frac{-\ln 3}{20}$ into ①: $\Rightarrow A = 90$

$$\therefore \boxed{T = 90e^{\frac{-(\ln 3)t}{20}} + 20}$$

9

[8]

(c) What is the initial temperature of the slab?

1m

$T(0) = 90 + 20$

$= \underline{\underline{110^\circ C}}$ ✓

[1]

(d) How long will it take for the temperature of the slab to drop to within 5° C of its final temperature?

2m

$T = 25$

$\therefore 90e^{-\frac{(\ln 3)t}{20}} = 5$

$t = \underline{\underline{52.6 \text{ minutes}}}$

✓ T = 25
✓ Solve for t

[2]

3

4. **[10 marks]**

A particle moves in simple harmonic motion in a straight line with a period of 5 seconds and amplitude of 4 metres. Initially the particle is 1 metre from its equilibrium point and is moving towards it. Determine:

(a) its distance from its equilibrium point after 3 seconds, to the nearest centimetre.

5m

$$x = 4\cos\left(\frac{2\pi t}{5} + \alpha\right)$$

✓ Amplitude
✓ trigonometric function
✓ Period
✓ phase shift
✓ distance at $t=3$

sub $t=0$, $x=1$: $1 = 4\cos\alpha$

$\cos\alpha = \frac{1}{4}$

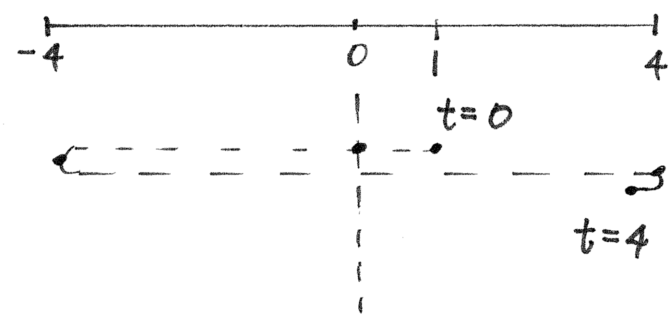
$\alpha = 1.318\ldots$

$$\therefore \quad x = 4\cos\left(\frac{2\pi t}{5} + 1.318\ldots\right)$$

when $t=3$, $x = 4\cos\left(\frac{6\pi}{5} + 1.318\ldots\right)$

$x = 1.4674$ m

$\approx \underline{\underline{1.47 \text{ m}}}$ (nearest centimetre)



5

[5]

(b) the total distance travelled in the first 4 seconds, to the nearest centimetre.

5m

when $x = 0$, $0 = 4\cos\left(\frac{2\pi t}{5} + 1.318\ldots\right)$

$\therefore t = 0.201$ sec

when $t = 4$, $x = 4\cos\left(\frac{8\pi}{5} + 1.318\ldots\right)$

$x = 3.9924$ m

when $x = 4$: $t = 0.201 + 3(1.25)$

$= 3.951$ sec

$\therefore$ Distance to $t = 3.951$ sec :

$= 1 + 3(4)$

$= 13$ m

since $x(4) = 3.9924$ m

[5]

$\therefore$ total distance travelled in 4 seconds

$= 13 + (4 - 3.9924)$

$= 13.008$ m

$\approx \underline{\underline{13.01 \text{ m}}}$ (nearest centimetre)

✓ t value for $x=0$
✓ x value for $t=4$
✓ t value for $x=4$
✓ total dist. travelled as a sum
✓ Final answer

5

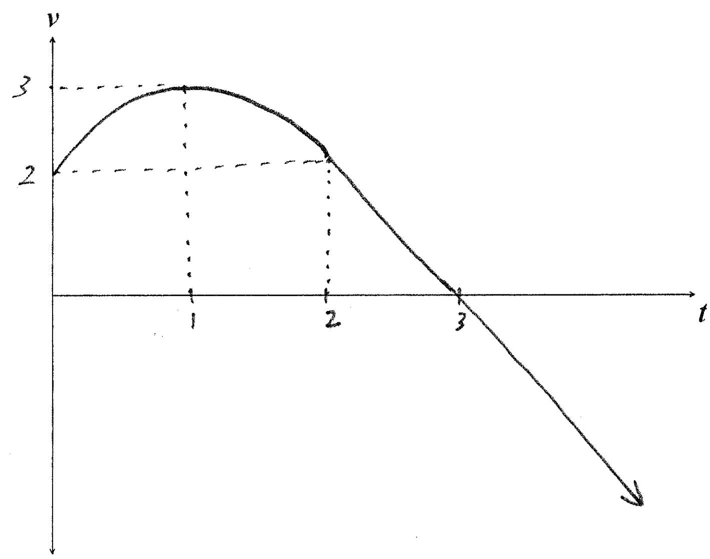
5. **[12 marks]** $\therefore\ x(0) = 0$

A particle is first observed at time $t = 0$ and its position at this point is taken as its initial position. The particle moves in a straight line so that its velocity, v, at time t is given by:

$$v = \begin{cases} 3 - (t-1)^2 & \textit{for } 0 \le t \le 2 \\ 6 - 2t & \textit{for } t > 2 \end{cases}$$

(a) On the axes below, sketch the velocity-time graph for $t \ge 0$.

3m



✓ $v = 3 - (t-1)^2$ correct

✓ $v = 6 - 2t$ correct

✓ essential points displayed.

[3]

(b) Determine the distance travelled by the particle from its initial position until it first comes to rest.

4m

$$\text{Distance} = \int_0^3 v(t)\, dt$$

$$= \int_0^2 3 - (t-1)^2\, dt + \int_2^3 (6-2t)\, dt$$

$$= \underline{\underline{\frac{19}{3}}} \text{ units}$$

✓ Accumulation function

✓✓ piecewise integration

✓ correct distance

[4]

7

(c) Determine the time, correct to two decimal places, when the particle returns to its original position.

3m

$$\int_0^3 v(t)\,dt + \int_3^T v(t)\,dt = 0$$

✓ integral with unknown upper limit.

$$\int_0^2 3-(t-1)^2\,dt + \int_2^T (6-2t)\,dt = 0$$

✓ piecewise defined integral.

✓ Solve for unknown.

$$T = \underline{\underline{5.52}} \text{ units}$$

[3]

(d) Calculate the acceleration of the particle when $t = 2$.

2m

$$a = \begin{cases} -2t+2 & , \; 0 \le t \le 2 \\ -2 & , \; t > 2 \end{cases}$$

✓ limit of acceleration from either side of $t=2$

✓ Correct answer.

as $t \to 2^-$, $a = -2$

as $t \to 2^+$, $a = -2$

[2]

$\therefore$ acceleration at $t=2$ is $\underline{\underline{-2}}$ units / (unit of time)^2

5

END OF TEST